REPORT RESUMES

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ACTION WITH WHOLE NUMBERS.

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THIS BOOKLET, ONE OF A SERIES, HAS BEEN DEVELOPED FOR THE PROJECT, A PROGRAM FOR MATHEMATICALLY UNDERDEVELOPED PUPILS. A PROJECT TEAM, INCLUDING INSERVICE TEACHERS, IS BEING USED TO WRITE AND DEVELOP THE MATERIALS FOR THIS PROGRAM. THE MATERIALS DEVELOPED IN THIS BOOKLET INCLUDE (1) ADDITION AND SUBTRACTION WITH WHOLE NUMBERS, (2) PATTERNS AND PROCEDURES IN MULTIPLICATION, AND (3) DIVISION AS A PROCESS OF FINDING A MISSING FACTOR. ACCOMPANYING THESE BOOKLETS WILL BE A "TEACHING STRATEGY BOOKLET" WHICH WILL INCLUDE A DESCRIPTION OF TEACHER TECHNIQUES, METHODS, SUGGESTED SEQUENCES, ACADEMIC GAMES, AND SUGGESTED VISUAL MATERIALS. (RP)

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ACTION

WITH

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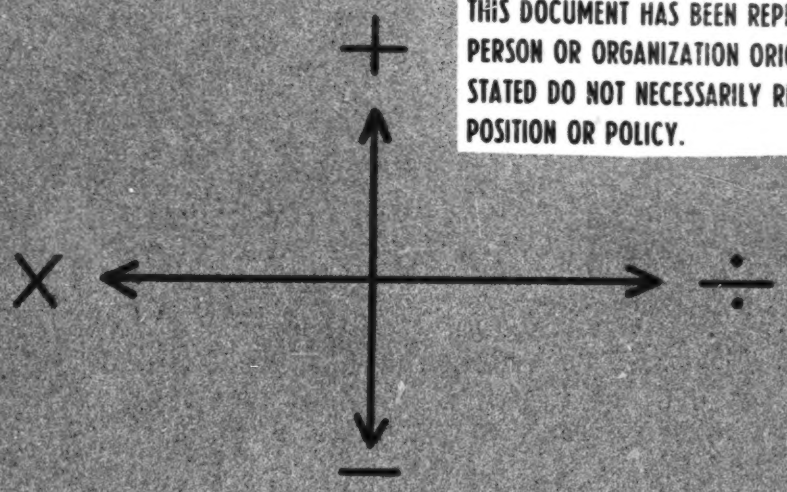

WHOLE

NUMBERS

SE 004 519

ESEA Title III

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ACTION WITH WHOLE NUMBERS

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ADDITION AND SUBTRACTION—Action With Whole Numbers

The Number Idea

For a long time you have known how to count. How old were you when you first counted to ten? Actually you have used these numbers for a long time without really paying any attention to them. Suppose we look at some ideas which will help you improve your understanding of numbers.

First you should be able to distinguish between a number and its name. "Jack" is a boy's name. The name "Jack" which appears on this page is not a boy. The name does not have two arms, two legs, two eyes, etc. and does not eat, sleep, and breathe. In a similar manner numbers and names of numbers are not the same. You will never see a number. Those "things" which appear on paper are really names of numbers, even though most people call them numbers.

Let us write the names of some numbers. The number 2 for example:

1 + 1

0 + 2

2 + 0

How about the number 3?

1 + 1 + 1

1 + 2

2 + 1

3 + 0

0 + 3

The number 4:

1 + 1 + 1 + 1

2 + 2

3 + 1

1 + 3

0 + 4

4 + 0

Activities

Rename each of the numbers named below. Give 5 different names for each number.

1. 5

2. 6

3. 7

4. 8

5. 9

Suppose you want to add 9 and 6, and you have forgotten the answer. Can you think of a way to do it? How about renaming the number 6?

$$\begin{aligned} 9 + 6 &= 9 + 1 + 5 \\ &= 10 + 5 \\ &= 15 \end{aligned}$$

Activities

Find the sum (add) in each of the following problems by renaming one of the addends as in the preceding example.

1. 9 + 2 =

2. 9 + 3 =

3. 9 + 4 =

4. 9 + 5 =

5. 9 + 6 =

6. 9 + 7 =

7. 9 + 8 =

8. 9 + 9 =

9. 8 + 3 =

10. 8 + 4 =

11. 8 + 5 =

12. 8 + 6 =

13. 8 + 7 =

14. 8 + 8 =

15. 8 + 9 = (Have you already done this problem?)

16. 7 + 4 =

17. 7 + 5 =

18. 7 + 6 =

19. 7 + 7 =

20. 7 + 8 = (Where did you do this one?)

21. 7 + 9 = (Sorry you have done this before too.)

22. 6 + 5 =

23. 6 + 6 =

24. 6 + 7 = (Oops!)

25. 6 + 8 = (Again !!)

26. 6 + 9 = (Again!!)

Our Numeration System

Two reasons why the numeration system which we use is so very good are:

(1) it is a place value system; and

(2) the zero concept.

Because of these, it is possible to rename numbers in such a way that computing is very easy.

Examples:

1. 324 means 300 + 20 + 4;
   We say 3 one hundreds plus 2 tens plus 4 units.
2. It could also mean 300 + 10 + 14;
   3 one hundreds plus 1 ten plus 14 units.
3. or 324 = 320 + 4;
   32 tens plus 4 units.
4. or 324 = 310 + 14;
   31 tens plus 14 units.
5. or 324 = 200 + 120 + 4.
6. It helps to practice renaming numbers. (use "+" rather than "and.")

453 = 400 + 50 + 3
400 + 40 + 13
300 + 140 + 13

$$\begin{array}{rcl} 7.\quad 516 & = & 500 + 10 + 6 \\ \underline{+\ 221} & = & \underline{200 + 20 + 1} \\ & & 700 + 30 + 7 = 737 \end{array}$$

Activities

Rename each of the numbers named in problems 1 - 10 using expanded notation. The first is an example

1. 572 = 500 + 70 + 2
2. 95
3. 215
4. 43
5. 104
6. 340
7. 999
8. 2,346
9. 1,004
10. 15,706

Find three ways, using only addition, to write:

11. 39
12. 42
13. 148
14. 122
15. 2,076

Solve the following addition problems by renaming as in example 7.

$$16.\quad \begin{array}{r} 82 \\ \underline{+\ 75} \end{array} \qquad\qquad 17.\quad \begin{array}{r} 72 \\ \underline{+\ 28} \end{array}$$

18. 209
+ 615

19. 89
+ 91

20. 72
35
+ 38

21. 26
67
+ 14

22. 486
+ 514

23. 3008
+ 6029

24. 1486
+ 614

25. 5088
+ 3203

26. 1276
4950
+ 6057

27. 65243
10750
+ 69374

28. 931 + 3908 =

29. $39.08 + $41.02 =

30. $19.75 + $315.25 =

Inverses

This morning you "put on your shoes," tonight you will "take off your shoes." These two actions are called inverses of each other.

Activities

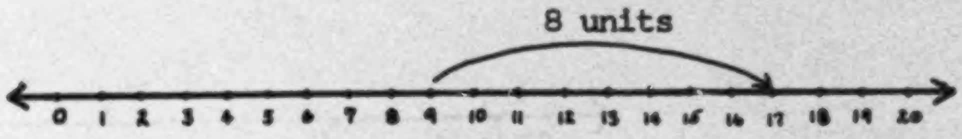
Some actions are described here. Give the inverse of each. (The first is completed for you.)

	Action	Inverse Action
1.	Open your book	Close your book
2.	Stand up	
3.	Face north	
4.	Look up	
5.	Run upstairs	
6.	Turn left	
7.	Look down	
8.	Walk east	
9.	Add 5	
10.	Subtract 100	

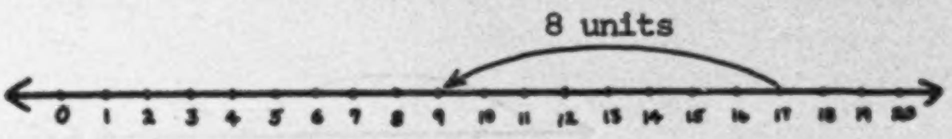
What does the idea of inverses have to do with mathematics? The mathematical operation subtraction is the inverse of addition.

Examples:

1. 9 + 8 = 17


8 units
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

2. 17 - 8 = 9

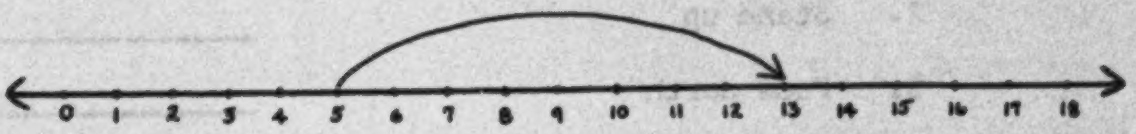

8 units
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

In the first example you begin at 9 and move 8 spaces to the right on the number line. In this subtraction example you begin at 17 and move 8 spaces to the left on the number line. Do you see why subtraction is said to be the inverse operation with respect to addition. Addition is also called the inverse operation with respect to subtraction.

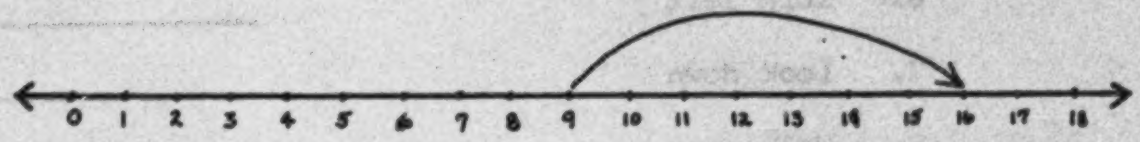
Activities

In each of the problems 1 - 10 write the addition or subtraction problem which is illustrated by the picture of the number line.

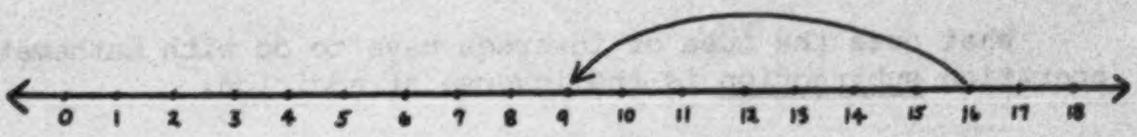
1. 5 + ___ = ___


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18

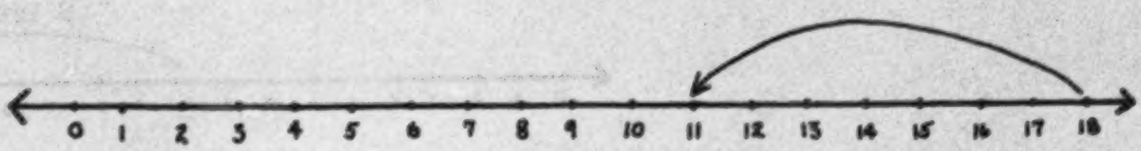
2. ______________________


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18

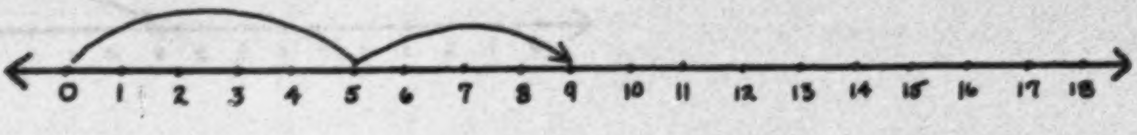
3. ______________________


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18

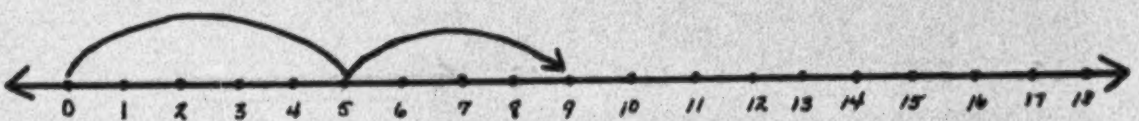
4. ______________________


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18

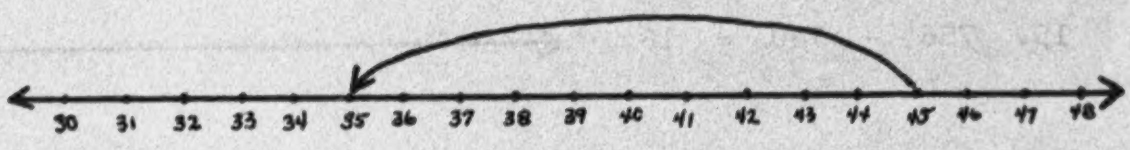
5. ______________________


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18

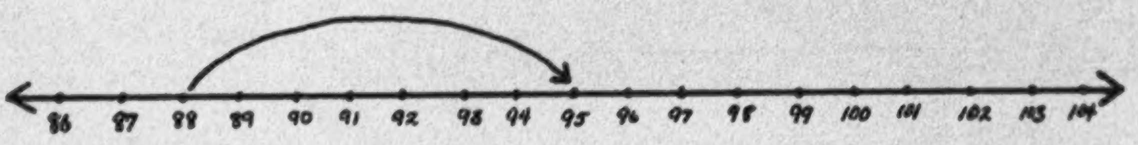
6. ______________________


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18

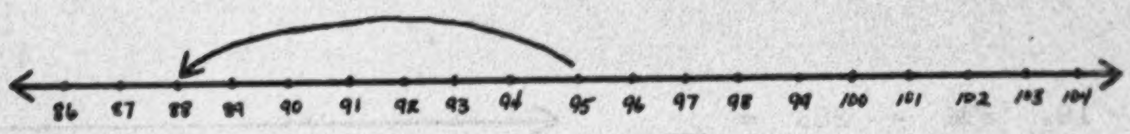
7. ______________________


30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48

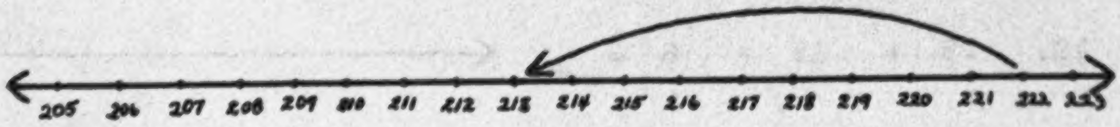
8. ______________________


86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104

9. ______________________


86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104

10. ______________________


205 206 207 208 209 210 211 212 213 214 215 216 217 218 219 220 221 222 223

Use the pictures of the number lines which follow to solve problems 11 - 20.

11. 12 + 9 = 21

12. 21 - 9 = 12

13. 256 - 243 = 13

14. 25 + 9 + 11 = 45

15. 756 - 740 = 16

16. 86 + 13 = __

17. 99 - 13 = __

18. 294 - 283 = __

19. 25 + 15 + 6 = __

20. 763 - 745 = __

As has already been stated, the renaming of numbers often simplifies subtraction.

Examples:

1.
$$\begin{array}{rcl} 2936 & = & 2000 + 900 + 30 + 6 \\ \underline{-\ 1421} & = & \underline{1000 - 400 - 20 - 1} \\ & & 1000 + 500 + 10 + 5 \ = \ 1515 \end{array}$$

2.
$$\begin{array}{rcrcrcrl} 133 & & 100 + 30 + 3 & = & 100 + 20 + 13 & = & 60 + 60 + 13 & \\ \underline{-\ \ 69} & = & \underline{-\ 60 - 9} & = & \underline{-\ 60 -\ 9} & = & \underline{-\ 60 -\ 9} & \\ & & & & & & 60 + \ 0 + \ 4 & = \ 64 \end{array}$$

Activities

Solve the following addition and subtraction problems by renaming. Use expanded notation.

1.
$$\begin{array}{r} 65 \\ \underline{-\ \ 43} \end{array}$$

2.
$$\begin{array}{r} 86 \\ \underline{-\ \ 52} \end{array}$$

3.
$$\begin{array}{r} 85 \\ \underline{+\ \ 76} \end{array}$$

4. $\begin{array}{r} 358 \\ -\ \ 69 \\ \hline \end{array}$

5. $\begin{array}{r} 256 \\ -\ \ 63 \\ \hline \end{array}$

6. $\begin{array}{r} 9750 \\ -\ 6029 \\ \hline \end{array}$

7. $\begin{array}{r} 731 \\ +\ 645 \\ \hline \end{array}$

8. $\begin{array}{r} 23 \\ 35 \\ +\ \ 67 \\ \hline \end{array}$

9. $\begin{array}{r} 19 \\ -\ \ 19 \\ \hline \end{array}$

10. $\begin{array}{r} 2564 \\ -\ 1675 \\ \hline \end{array}$

```
11.     3569
        6042
        9406
      + 2631
```

```
12.      492
      -  273
```

```
13.    59864
      -  242
```

```
14.    75984
       43256
       30091
     + 25070
```

```
15.      752
      -  674
```

```
16.      895
      -  697
```

17. 253 + 12 + 46 =

18. 5692 - 3461 =

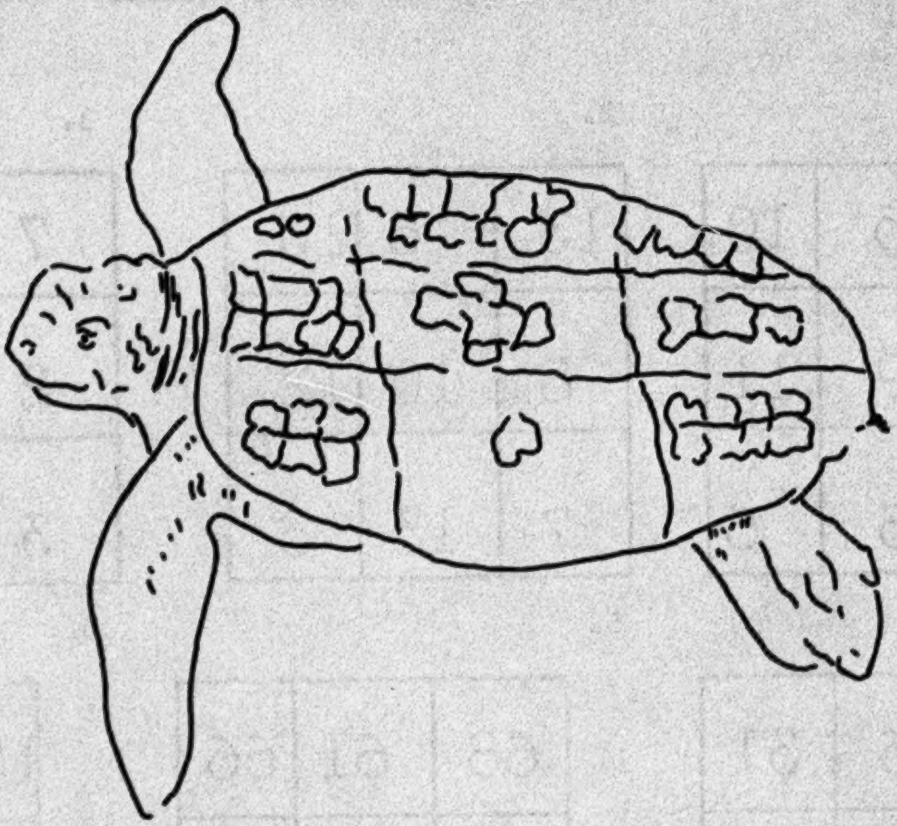
19. 3256 - 378 =

20. 25 + 32 + 7 + 49 =

Magic Squares

Picture yourself on the banks of the Yellow River. It is the year 2200 B. C. (of course you don't know it's B. C. You don't even know it's the year 2200 because a system of recording years hasn't been decided). You are Emperor Yu. You have not been well and all your healers have not been able to help you. As you look at the muddy water, you suddenly feel better and then—a large turtle crawls out of the water. This turtle is like no other turtle you have ever seen. The markings on its back are very strange. Surely this must be a sign from the gods, but what does it mean?

On page 15 is a drawing of the turtle. Since you are Emperor, everyone expects you to figure out this sign. What can you see about this turtle? Did you ever see a turtle with markings like this. You must show your people something different about this turtle because you have already proclaimed that it caused you to feel better. What do you see? Remember, you have to find something. Is there any pattern to the markings?



2	9	4
7	5	3
6	1	8

Just such a turtle is supposed to have crawled out of the Yellow River. This fact is recorded in Y.I.H.--King which is said to be the oldest book in the world. On the turtle's back is the first known "MAGIC SQUARE." It was used as a charm to drive away disease and evil. A "magic square" is a square array of numbers such that all columns, rows, and diagonals have the same sum.

Activities

Which of the following square arrays are magic squares? Write yes or no.

1.

12	5	10
7	9	11
8	13	6

2.

13	6	10
8	10	12
7	13	8

3.

7	0	5
2	4	6
3	8	1

4.

33	26	31
28	30	32
29	35	27

5.

68	61	66
63	65	67
64	69	62

6.

153	146	151
148	150	152
149	159	147

7.

56	82	81	59
75	65	66	72
67	73	74	64
80	58	57	83

8.

120	146	145	123
139	129	130	136
131	137	138	128
144	122	121	147

9.

46	72	71	49
65	55	56	62
57	64	63	54
70	48	47	73

10.

132	158	157	135
151	141	142	148
143	149	150	140
159	134	133	156

11.

	6	
	10	12
9	14	7

12.

8		
	5	7
4		2

13.

9		
	6	8
5		3

14.

30		17	
19			22
23	21		26
18	28		15

15.

20			17
	15	14	
	11	10	16
8		19	5

16.

		39
	38	
37		35

17.

58	51	
53		
54		

18.

238		
233	235	237

19.

40	66		43
59	49		
	57	58	
64			61

20.

120	146		123
139			
	137		
144	122	121	147

TO MAKE A MAGIC SQUARE

3 by 3 "scheme"

Any 3 by 3 consecutive array Position in array		3 by 3 magic square Position in array
	goes to	
1	→	2
2	→	9
3	→	4
4	→	7
5	→	5
6	→	3
7	→	6
8	→	1
9	→	8

Consecutive Array

1 1	2 2	3 3
4 4	5 5	6 6
7 7	8 8	9 9

Magic Square

1 8	2 1	3 6
4 3	5 5	6 1
7 4	8 9	9 2

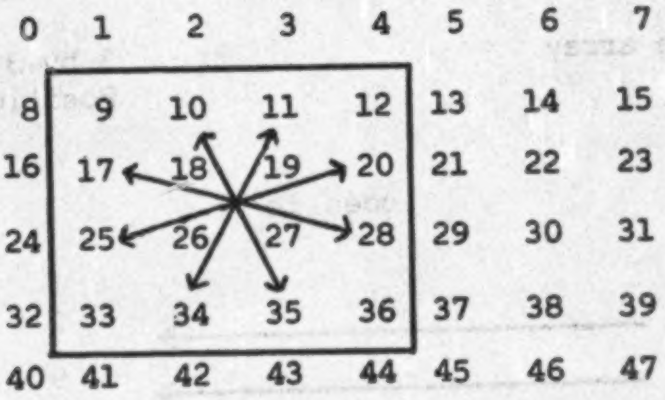
Consecutive Array

1 13	2 14	3 15
4 16	5 17	6 18
7 19	8 20	9 21

Magic Square

1 20	2 13	3 18
4 15	5 17	6 19
7 16	8 21	9 14

4 by 4 "scheme"


0 1 2 3 4 5 6 7
8 9 10 11 12 13 14 15
16 17 18 19 20 21 22 23
24 25 26 27 28 29 30 31
32 33 34 35 36 37 38 39
40 41 42 43 44 45 46 47

To make a magic square exchange positions as indicated above.

Before Exchange

9	10	11	12
17	18	19	20
25	26	27	28
33	34	35	36

After Exchange

9	35	34	12
28	18	19	25
20	26	27	17
33	11	10	36

Optional 4 by 4 "scheme"

0	1	2	3	4	5	6	7
8	9	⑩	⑪	12	13	14	15
16	⑰	18	19	⑳	21	22	23
24	㉕	26	27	㉘	29	30	31
32	33	㉞	㉟	36	37	38	39
40	41	42	43	44	45	46	47

Instead of exchanging the numbers as above, add the largest and the smallest numbers in the array (9 + 36 = 45). Subtract each number circled from 45 and replace circled numbers with the difference. (45 - 10 = 35, thus 10 is replaced by 35.) Do the same for all other circled numbers.

Exploring Other Procedures

In some countries the number 13 is associated with bad luck, and in other countries it is associated with good luck. We write and read from left to right, yet people from some countries in the far east write and read from "top to bottom" and "right to left." We do addition and subtraction from "right to left." Many early civilizations did their addition and subtraction from "left to right." We learn a particular way to do something and in many cases are unaware that it could be done differently or that people in other places actually use a different way.

"Left to Right" Addition

Suppose we examine some typical problems using both addition and subtraction and see the ways in which we "could" arrive at an answer. First consider the addition problem below, going from "left to right."

Example:

$$\begin{array}{r} 426 \\ \underline{+\ \ 865} \\ 1200 \\ 80 \\ \underline{+\ \ \ 11} \\ 1291 \end{array}$$

Step I: 400 + 800

Step II: 20 + 60

Step III: 6 + 5

Step IV: Add column from left to right.

Activities

Complete these addition problems using "left to right" addition. Look again at the example above.

1. $\begin{array}{r} 853 \\ \underline{+\ \ 844} \end{array}$

2. $\begin{array}{r} 798 \\ \underline{+\ \ 849} \end{array}$

3. $\begin{array}{r} 654 \\ \underline{+\ \ 986} \end{array}$

4. 9654
+ 7482

5. 7236
+ 2576

6. 274
605
+ 329

Now shorten this process by not putting in the zeros. Problem 7 is an example.

7. 853
+ 844
16
9
7
1697

8. 798
+ 849

9. 654
+ 986

10. 9654
+ 7482

11. 7236
+ 2576

12. 274
605
+ 329

13. 259
+ 643

14. 763
+ 251

15. 6053
+ 9816

Ancient Scratch Method

You are now ready to use an ancient procedure for addition called the scratch method. Draw a line through numbers as you add them. To show it in a way that is easy to see, let's do it first in steps.

Step I:

$$\begin{array}{r} \cancel{8}64 \\ +\ \cancel{3}78 \\ \hline 11 \end{array}$$

Step II:

$$\begin{array}{r} \cancel{8}\cancel{6}4 \\ +\ \cancel{3}\cancel{7}8 \\ \hline 11 \\ 13 \end{array}$$

Step III:

$$\begin{array}{r} \cancel{8}\cancel{6}\cancel{4} \\ +\ \cancel{3}\cancel{7}\cancel{8} \\ \hline 11 \\ 13 \\ 12 \\ \hline 1242 \end{array}$$

Complete: 1242

Activities

Try these problems using the scratch method.

1. $\begin{array}{r} 657 \\ +\ 453 \\ \hline \end{array}$

2. $\begin{array}{r} 385 \\ +\ 926 \\ \hline \end{array}$

3. $\begin{array}{r} 987 \\ +\ 987 \\ \hline \end{array}$

4. $\begin{array}{r} 678 \\ +\ 933 \\ \hline \end{array}$

5. $\begin{array}{r} 547 \\ +\ 658 \\ \hline \end{array}$

6. $\begin{array}{r} 734 \\ +\ 697 \\ \hline \end{array}$

How do you like this way of doing addition? Is it similar to our way? Now that you have control of the situation does it matter, as far as the answer is concerned, whether you begin on the left, right, or in the middle. Maybe we should try one by beginning in the middle.

$$\begin{array}{r l}
6\cancel{9}4 & \\
+\quad 9\cancel{8}7 & \\ \hline
170 & (\ 90 + \ 80)\text{---Step I} \\
1500 & (600 + 900)\text{---Step II} \\
11 & (\ \ 4 + \ \ 7)\text{---Step III} \\ \hline
1681 & (\text{ Complete })
\end{array}$$

Activities

1. What ideas about numbers do you have to know in order to begin wherever you please?

2. Which is faster—this way or the regular way we add? Give reasons for the answer above.

3. Why do you think some early civilizations would add and subtract from left to right? Why do you think we add from "right to left?"

Do each of the addition problems our regular way and check the answer by doing it a different way.

	Regular	Different
4.	96 + 58	96 + 58
5.	959 + 876	959 + 876
6.	8659 + 9877	8659 + 9877
7.	573 + 615	573 + 615
8.	27 56 + 35	27 56 + 35
9.	984 + 656	984 + 656
10.	3942 + 643	3942 + 643

Compared to the procedure we use today, some ancient civilizations developed unusual ways to find the difference between two numbers. Some of these procedures worked from "left to right." The "scratch method" will serve as an illustration. Again let's do it in simple steps to get the idea. Suppose we are to subtract 38 from 72. We write the 38 under the 72 and find it is necessary to regroup (borrow) or rename 72. However, if we begin at the left, do we do it this same way?

$$\begin{array}{r} 72 \\ \underline{-\ 38} \\ \cancel{4}4 \\ 3 \end{array}$$

Step I: Subtract 3 from 7

Step II: You can't subtract 8 from 2 and get a whole number; therefore, you take one of the 4 (tens) and "carry" it up and add it to 2 giving 12 ones. Now 12 - 8 = 4. Your answer is 34.

Activities

Try the scratch subtraction method on these problems.

1. $\begin{array}{r} 62 \\ \underline{-\ 48} \end{array}$

2. $\begin{array}{r} 96 \\ \underline{-\ 37} \end{array}$

3. $\begin{array}{r} 51 \\ \underline{-\ 24} \end{array}$

4. $\begin{array}{r} 73 \\ \underline{-\ 46} \end{array}$

5. $\begin{array}{r} 86 \\ \underline{-\ 59} \end{array}$

Now suppose the problem is a little harder, say:

$$\begin{array}{r} 321 \\ \underline{-\ 154} \\ \cancel{2}\cancel{7}7 \\ 16 \end{array}$$

(answer 167)

Use the same idea and go from the hundreds to the ones "carrying" back up when necessary.

The examples we worked are actually not exactly the way the "scratch method" worked. They placed the answer above the problem. See if you can figure out the one below.

$$\begin{array}{l} 36 \\ \cancel{4}\cancel{7}7 \\ \cancel{5}2\cancel{6} \\ \cancel{1}\cancel{5}\cancel{9} \end{array}$$

(answer 367)

Activities

Try these using the above idea.

1. $\begin{array}{r} 654 \\ \underline{-\ 266} \end{array}$

2. $\begin{array}{r} 614 \\ \underline{-\ 148} \end{array}$

3. $\begin{array}{r} 503 \\ \underline{-\ 386} \end{array}$

4. $\begin{array}{r} 523 \\ \underline{-\ 154} \end{array}$

5. $\begin{array}{r} 893 \\ \underline{-\ 121} \end{array}$

Another name of an older procedure is called the "Equals—Addition Method" and another is the "Complement Method." Maybe someone in the class could look up these procedures and give a class report.

Activities

Set up a subtraction problem from each of the verbal problems and solve.

1. A football field is 360 feet long, including the end zones. The width of the field is 160 feet. How much more is the length than the width?

2. There are presently 206 radio stations and 18 television stations in Florida. The number of television stations is __________ less than the number of radio stations.

3. The first gasoline engine for boats was perfected around 1900. How many years has it been since?

4. A professional basketball team won 52 games. Another team in the league was not as fortunate and won only 14 games. What was the difference of wins?

5. At the Daytona International Speedway there were, during one racing year, a total of 763 Corvettes entered in races. Pontiac G.T.O.'s numbered 250. How many more Corvettes were there?

6. Joe has a surfboard that is 10 feet long. Bill's board is only 8 feet. How much shorter is Bill's board?

7. The U. S. Marines made their first landing in 1776 on the Bahama Islands during the Revolutionary War. How many years has it been since their first landing?

8. At one time the population of Golfview in Palm Beach County was 84. In 1960 it was 131. How much increase was there in population?

There are many ways of asking for the difference of two numbers. Suppose we have the problem: 82 - 14 = 68. Place the 82 and the 14 in the proper space:

9. ____________________ take away ____________________ = ____________________

10. ____________________ less ____________________ = ____________________

11. ____________________ exceeds ____________________ = ____________________

12. ____________________ is ____________________ = ____________________

13. ____________________ is ____________________ = ____________________

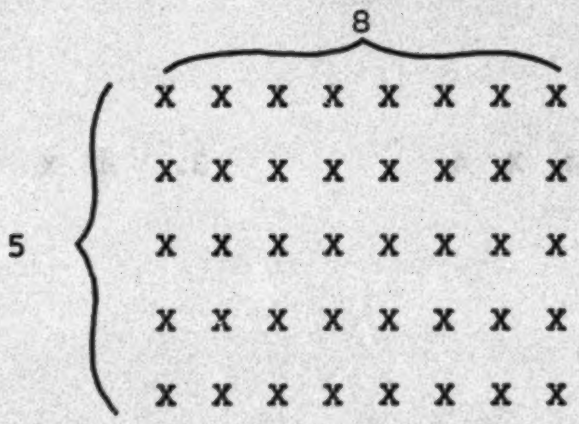
14. Can you think of other ways of asking this question?

MULTIPLICATION—Patterns and Procedures

One way to represent the product of two numbers is by using arrays.

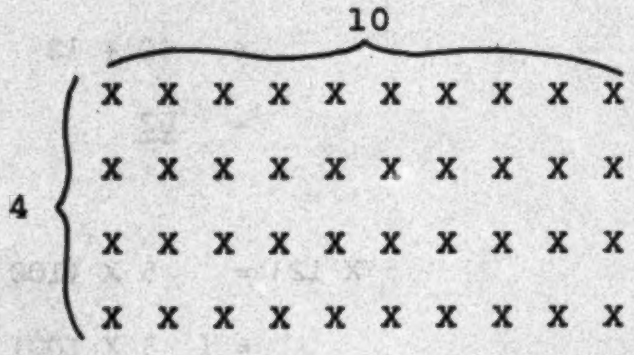
The array below suggests the product: 5 X 8 = 40

That is, 5 rows with 8 in each row.



Can these 40 X's be arranged in a different rectangular array?

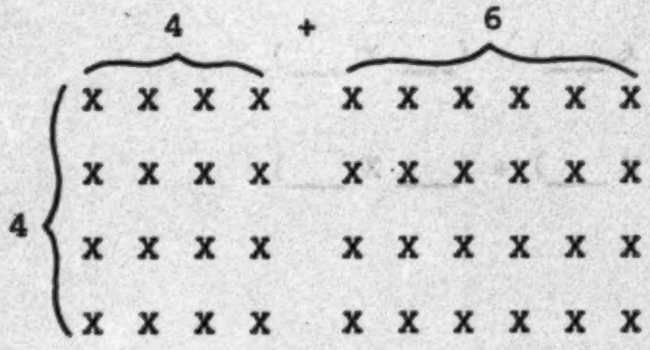
You are right! Another rearrangement is shown below.



which illustrates: 4 X 10 = 40

Examine the 4 X 10 array below, it is shown as: (4 X 4) + (4 X 6) =

(4 X 4) + (4 X 6)



Looking at the array (last one on page 29), we can see that:

$$4 \times (4 + 6) = (4 \times 4) + (4 \times 6)$$

The example above illustrates property of multiplication over addition.

Activities

Show as an array:

1. 8 X 2
2. 4 X 4
3. 4 X (2 X 2)

Lock at the two examples below showing different ways multiplication can be distributed over addition.

Example 1:

$$\begin{aligned} 4 \times 13 &= 4 \times (10 + 3) \\ &= (4 \times 10) + (4 \times 3) \\ &= 40 + 12 \\ &= \underline{52} \end{aligned}$$

Example 2:

$$\begin{aligned} 5 \times 123 &= 5 \times (100 + 20 + 3) \\ &= (5 \times 100) + 5 \times 20) + (5 \times 3) \\ &= 500 + 100 + 15 \\ &= \underline{615} \end{aligned}$$

Activities

Complete each of the following:

1. 4 X (5 + 3) = (___ X ___) + (___ X ___)
2. 8 X (20 + 3) = (___ X ___) + (___ X ___)

3. 10 X (100 + 20 + 3) = (___ X ___) + (___ X ___) + (___ X ___)

4. 12 X (200 + 40 + 6) = (___ X ___) + (___ X ___) + (___ X ___)

Some problems can be done much easier if they are rewritten and the distributive law applied. Below are some problems involving subtraction. The first two are complete. Use this idea to complete the others.

5. 9 X 19 = 9 X (20 − 1)
= (9 X 20) − (9 X 1)
= 180 − 9
= 171

6. 8 X 18 = 8 X (20 − 2)
= (8 X 20) − (8 X 2)
= 160 − 16
= 144

7. 8 X 39

8. 6 X 68

9. 7 X 19

10. 6 X 57

11. 9 X 29

12. 12 X 49

Supply the missing value that makes each statement true.

13. 6 X (9 + 4) = (6 X 9) + (___ X 4)

14. ___ X (8 + 5) = (10 X 8) + (10 X 5)

15. 5 X (12 + 4) = (5 X 12) + (5 X ___)

16. 9 X (___ + 10) = (9 X 12) + (9 X 10)

17. 4 X (7 + ___) = (4 X 7) + (4 X 8)

18. 8 X (___ + ___) = 8 X 8

Repeating Addends—Calculator Method

The common desk Calculator multiplies by repeating one factor as an addend the number of times indicated by the second factor. As an example:

52
x 4
52
52 four 52's
52
52
208 = four 52's

To multiply by 10, the carriage of the calculator moves over one space, for 100 it moves two spaces, and so forth. Example:

52
x 24
52
52
52 four 52's
52
520 ten 52's
520 ten 52's
1248 = twenty-four 52's

Activities

Use the Calculator Method to do the following:

1. 25 x 5
2. 31 x 32
3. 84 x 23

4. 56 x 21

5. 136 x 42

6. 186 x 31

7. 407 x 46

8. 927 x 51

9. 4232 x 23

10. 123 x 49

The calculator would use a "short-cut" in solving problem 10 of the preceding activities. It would solve the problem like this:

123		
x 49		
1230	ten 123's	
1230	ten 123's	
1230	ten 123's	fifty 123's
1230	ten 123's	
1230	ten 123's	
6150		
- 123	one 123	
6027 =	forty-nine 123's	

Activities

Solve the following multiplication problems by using the calculator method. Use the "short-cut" where it is applicable.

1. 35 x 29

2. 146 x 42

3. 352 x 39

4. 3569
x 99

5. 562
x 999

6. 2580
x 34

7. 925
x 49

8. 6579
x 19

9. 7256
x 58

10. 5243
x 99

Compare the Calculator Method to our procedure.

$$\begin{array}{r} 56 \\ \underline{22} \\ 112 \\ \underline{112} \\ 1232 \end{array}$$

Steps

1. 2 x 6 = 12. Put the 2 ones down and "carry" the 1 ten.
2. 2 x 5 = 10, 10 + 1 = 11. Put the 11 down.
   (Now we have repeated 52 as a factor 2 times.
3. Annex a zero when multiplying by the tens position and repeat the first two steps.
4. Add partial products.

Notice that the first partial product is:

2 x 56 = 112

and the second is:

20 x 56 = 1120

and the total is:

22 x 56 = 1232.

We can repeat an addend in ones up to 9 at a time, tens up to 90 at a time, hundreds up to 900 at a time and so forth, simply by knowing how to multiply by all single digits 1 through 9.

The calculator can do problems much faster simply because of the speed of electricity.

Activities

Do each of the problems by our regular method and the Calculator Method.

Regular	Calculator
1. $\begin{array}{r} 124 \\ \underline{\text{x}\ \ 34} \end{array}$	$\begin{array}{r} 124 \\ \underline{\text{x}\ \ 34} \end{array}$

2. 326
x 24

326
x 24

3. 567
x 41

567
x 41

A magic square in multiplication is a square array of numbers such that all rows, columns, and diagonals have the same product.
Are these magic squares?

4.

128	1	32
4	16	64
8	256	2

5.

4	1	16
16	4	1
1	16	4

6.

32	16	8
16	8	32
8	32	16

7.

2187	1	243
9	81	729
27	6561	3

Exploring Other Multiplication Procedures

Russian Peasant Method

Have you ever wondered if ancient civilizations did multiplication using the same procedure we use today? Actually some very interesting procedures were developed by different cultures. Perhaps one of the most unusual of these procedures is called the Halving and Doubling Method or the Russian Peasant Method. It is still used to some extent by peasants in some countries.

Here's "how" it works. Select two numbers to multiply. The example is (23 x 27):

Step 1: To multiply 23 by 27, choose one of the factors to halve and double the other.

Step 2: Continue halving one factor until you get 1 as an answer. Now double for each time you halved (when you halve an odd number disregard the fraction--for example halve 25 and use 12 not 12½.

Step 3: On the side you halved pick out the even numbers and cross out the corresponding double. The sum of the remaining doubles is the product.

	Halve	Double
	23	27
	11	54
	5	108
(even)	~~2~~	~~216~~
	1	432
		621

Then 23 x 27 = 621

Do you recall that in multiplication, that if factors are rearranged you will get the same product. For example:

2 x 5 = 5 x 2

or 3 x 6 x 4 = 4 x 3 x 6

This is called the Commutative Property of multiplication.

Is the Halving and Doubling Method Commutative? Can you choose either factor to halve and double the other and still get the same answer?

	Halve	Double
	27	23
	13	46
(even)	~~6~~	~~92~~
	3	184
	1	368
		621

Then 27 x 23 = 621, and it is Commutative.

Activities

Do the problems below using the halving and doubling method. Check your answer either by exchanging your factors to halve and double or by our multiplication method.

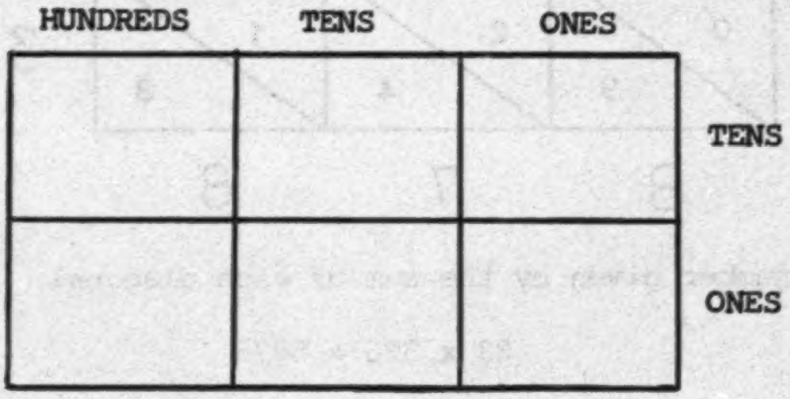
1. 25 x 31
2. 36 x 27
3. 56 x 81
4. 19 x 46
5. 53 x 21
6. 75 x 19

7. 63 x 71   8. 37 x 59   9. 35 x 123

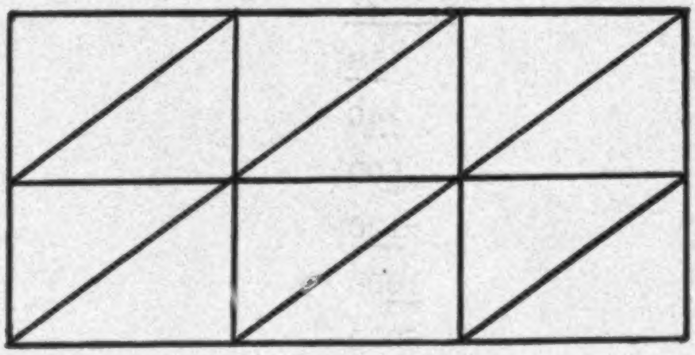
10. What are the advantages and disadvantages of the halving and doubling method?

Lattice Method

The next ancient procedure was no doubt named the Lattice Method because of the geometric figure that is used. Rows and columns of squares are drawn. The number of squares depends upon the number of digits in each factor. Place value for a three digit times a two digit number is shown below.



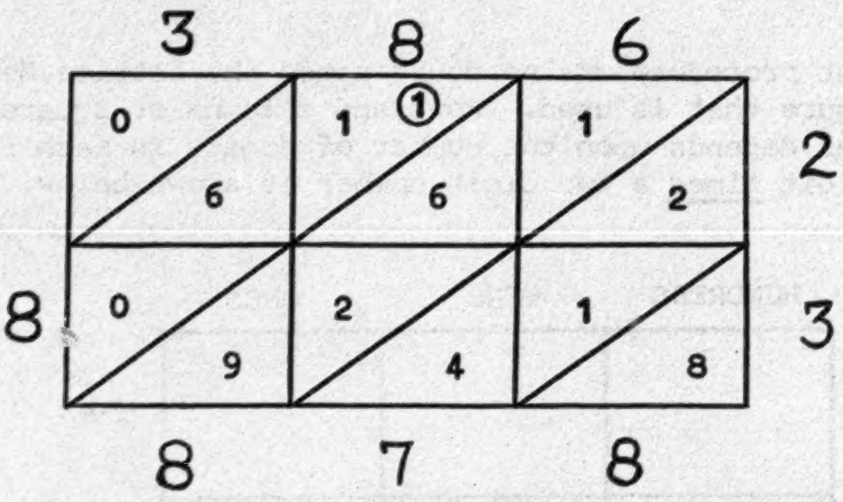
Diagonals are drawn in each square.



Now suppose we multiply: 386 X 23

(A) Start by using the right bottom square. If any two digits multiply and give a two digit product, place the last digit below the diagonal in the appropriate square and the first digit of the product above the diagonal.

(B) The digits in each diagonal are then added and "carry" numbers are placed in the diagonal on the previous page. "Carry" numbers are circled.



Then, reading the number given by the sum of each diagonal

23 x 386 = 8878

Below is the product in another form. Can you match up the partial products?

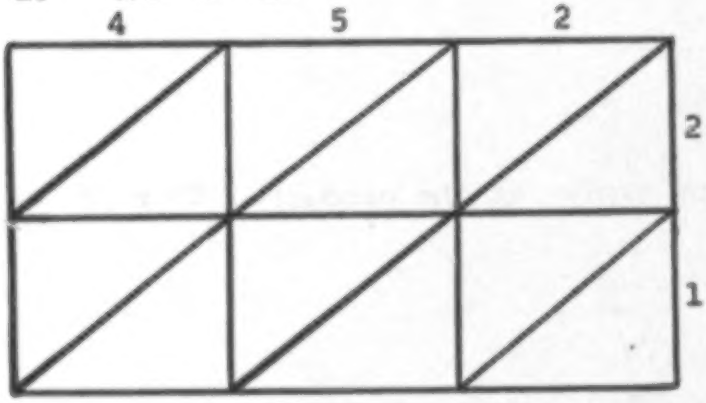
$$\begin{array}{r} 386 \\ \text{x}\quad 23 \\ \hline \underline{18} \\ \underline{24}0 \\ \underline{6}00 \\ \underline{12}0 \\ \underline{16}00 \\ \underline{6}000 \\ \hline 8878 \end{array}$$

Can you illustrate that the Commutative Property is true for this procedure?

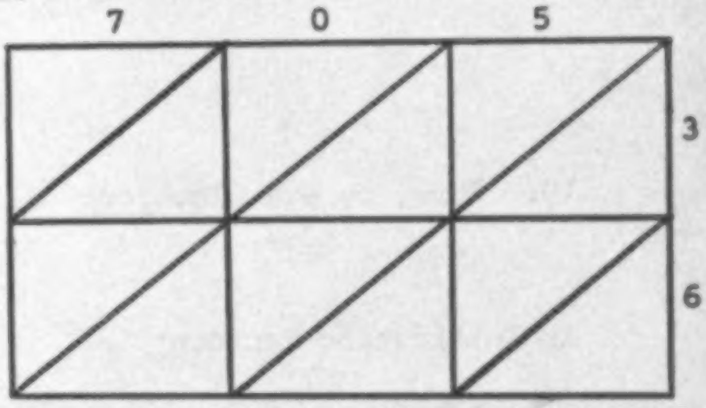
Activities

Use the Lattice Method to find the product in each of the problems 1 - 9.

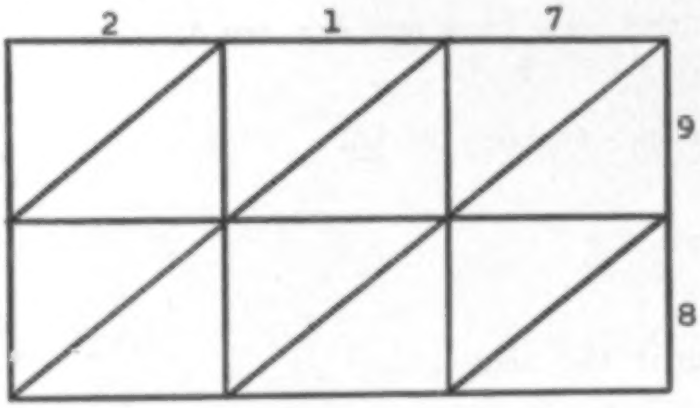
1. 452 X 21 =



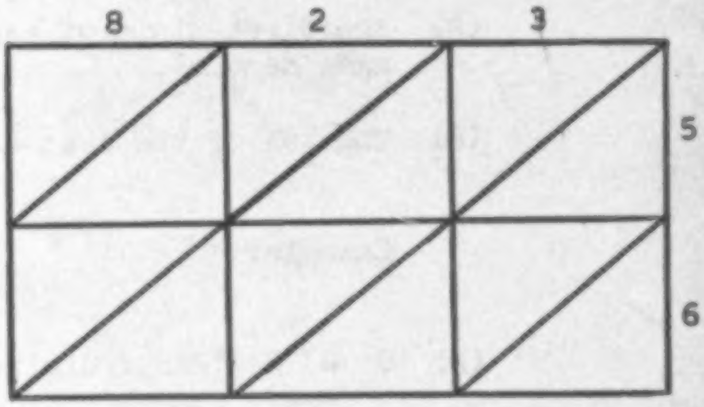
2. 36 X 705 =



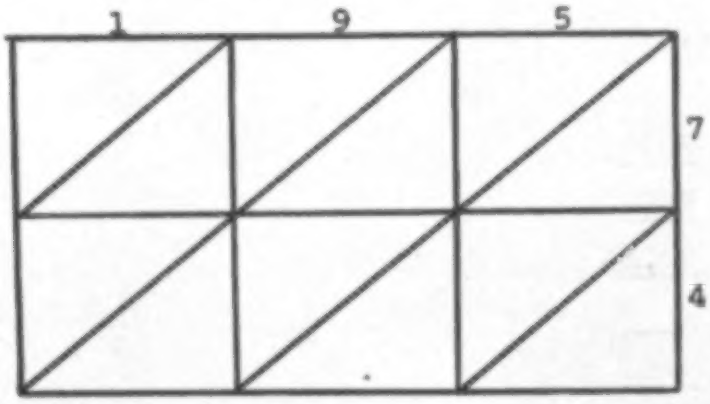
3. 217 X 98 =



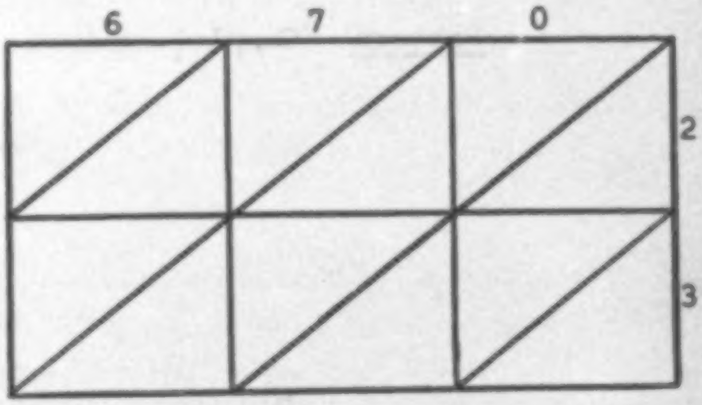
4. 823 X 56 =



5. 195 X 74 =



6. 670 X 23 =



7. 256 x 31 =

8.
```
   983
x   46
```

9.
```
   403
x   62
```

10. Show, by example, <u>four</u> different ways to arrive at the product: 57 x 23

An Interesting Pattern

Below is a pattern for multiplying two digit numbers which fit the following conditions:

(A) The first digit of each factor is the same (the numbers are in the same decade).

(B) The sum of the last digits, of the two factors is <u>10</u>.

Example: 63 x 67

(A) 6 = 6 (First digits of each factor the same.)

(B) 3 + 7 = 10 (Sums of the last digits is 10.)

<u>Pattern</u> (Complete the missing products.)

99 x 91 = <u>9009</u>
92 x 98 = <u>9016</u>
93 x 97 = <u>9021</u>
94 x 96 = ____
95 x 95 = ____
89 x 81 = <u>7209</u>
88 x 82 = <u>7216</u>
87 x 83 = ____
86 x 84 = ____
85 x 85 = ____
. .
. .
. .
(last two factors of pattern) 15 x 15 = <u>225</u>

Do you see this pattern? For these special products, notice that:

(A) The products of ones digits gives the last two digits of the answer.

For: 67 x 63 = ______21

(B) The first two digits of the answer are obtained by multiplying the first digits of either factor (they are the same) by one more than itself.

THEN: 63 x 67 = 4221

Last two digits of the answer: 3 x 7 = 21

First two digits of the answer: 6 x 7 = 42

Activities

Try out this idea for the following products.

1. 94 x 96 =
2. 15 x 15 =
3. 26 x 24 =
4. 21 x 29 =
5. 63 x 67 =
6. 47 x 43 =
7. 52 x 58 =
8. 36 x 34 =
9. 32 x 38 =
10. 41 x 49 =
11. 25 x 25 =
12. 16 x 14 =
13. 28 x 22 =
14. 23 x 27 =
15. 54 x 56 =
16. 35 x 35 =
17. 45 x 45 =
18. 95 x 95 =
19. 87 x 83 =

Another Multiplication Procedure

Examine the multiplication procedure used below. This procedure is used to multiply two numbers where each number is close to 100. The product of: 93 x 94 = ______ may be obtained in the following way.

(A) Subtract each of the factors from 100 and place the difference over to the right of each factor;

93 — 7
x 94 — 6

100 - 93 = 7
100 - 94 = 6

(B) Multiply the differences and this is the last two digits of your answer to (93 x 94);

6 x 7 = 42

(C) Subtract: 93 - 6 = 87 or 94 - 7 = 87
and this is the first two digits of your answer.

THEN:

$$\begin{tikzcd}
93 \arrow[r] & 7 \\
\times 94 \arrow[r] & \times 6 \arrow[ul] \\
\underline{87} & \underline{42}
\end{tikzcd}$$

Activities

Try out this idea on the following examples.

1. 96
x 97

2. 98
x 90

3. 91
x 92

4. 99
x 98

5. 95
x 97

6. Check the problems above using a different multiplication procedure. Are your answers the same?

Nine Patterns

Just for fun look at the nines facts given below. Discuss all patterns you observe.

9 x 1 = 9
9 x 2 = 18
9 x 3 = 27
9 x 4 = 36
9 x 5 = 45

9 x 6 = 54
9 x 7 = 63
9 x 8 = 72
9 x 9 = 81

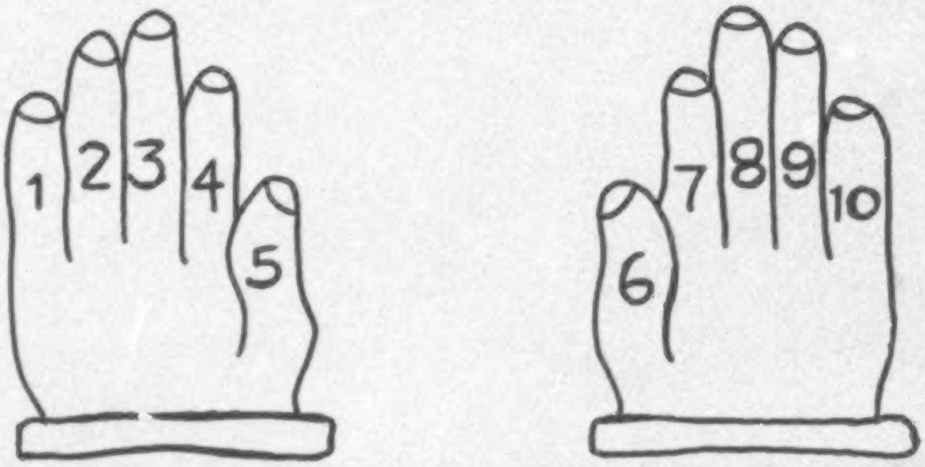
Did you find these:

(A) The sum of the digits in the product is 9.

(B) The answer (product) always begins with a number which is one less than the second factor.

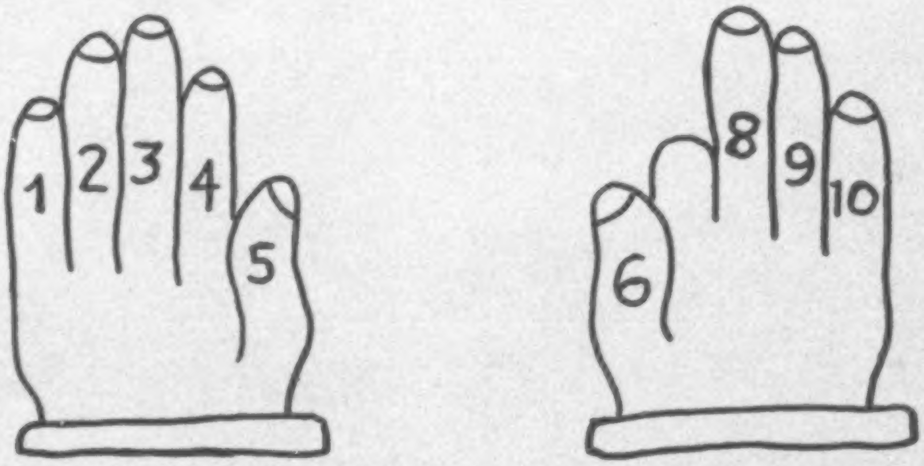
(C) The first column of digits in the answers increases by one while the second column decreases by one.

Just for fun, a person thinks of their fingers as being numbered from 1 to 10, counting from the left little finger (hands out flat).



To multiply 9 by any number by a number represented on a particular finger. Then turn under that finger (say by 7).

THEN:



How many fingers are up to the left of the 7th finger? Right! There are 6.

To the right of the 7th finger? Correct! There are 3. The answer is:

7 x 9 = 63.

Activities

Find the following products using your fingers.

1. 9 x 2 =

2. 9 x 3 =

3. 9 x 4 =

4. 9 x 5 =

5. 9 x 6 =

6. 9 x 8 =

7. 9 x 9 =

8. 9 x 10 =

DIVISION—Find a Missing Factor

Below is a group of X's.

X X X X X X X X X X X X X X X X X X X X X X X X

How many groups, 6 in each, do you have in the total group?
In mathematical terms the question is:

1. $24 \div 6 =$

2. or, $? \times 6 = 24$

Then our problem is a division problem, which actually, as indicated by step 2., is finding a missing factor. How is this question answered?

(A) We could group by sixes and count the groups.

(X X X X X X)	(X X X X X X)	(X X X X X X)	(X X X X X X)
1	2	3	4

Then the answer is 4.

(B) We could simply know that there are 4 sixes in 24 (from multiplication).

(C) We could subtract sixes out and count.

```
                         24
                       -  6      one 6
                         18
                       -  6      one 6
                         12
                       -  6      one 6
                          6
                       -  6      one 6
24 ÷ 6 = 4                0      four 6's
```

Then we count the number of sixes we subtracted out, and there are 4. Increase the difficulty of the problem, but use the same idea. How many times is 21 contained in 68.

$$\begin{array}{rl} 68 & \\ \underline{-\ 21} & \text{one } 21 \\ 47 & \\ \underline{-\ 21} & \text{one } 21 \\ 26 & \\ \underline{-\ 21} & \underline{\text{one } 21} \\ 5 & \text{three } 21\text{'s} \end{array}$$

68 ÷ 21 = 3 with a remainder of 5

THEN: there are 3 with a remainder of 5.

We could say: 68 + 21 = 3 with r = 5

or:

$$\begin{array}{r} 3 \\ 21\,\overline{)\,68} \\ \underline{63} \\ 5 \end{array}$$

Do you recognize the last form?

In multiplication, recall the Calculator Method that was developed. Multiplication was performed by repeated addition. Since division is the inverse of multiplication, it is performed by repeated subtractions. Suppose you used a desk calculator to do a division problem. As an example:

483 ÷ 21

The question is: How many 21's are contained in 483? The Calculator will do the division problem in the following way.

$$\begin{array}{rrl} 483 & & \\ \underline{-\ 21} & 1 & \text{Step I} \\ 273 & & \\ \underline{-\ 21} & 1 & \text{Step II} \\ 63 & & \\ \underline{-\ 21} & 1 & \text{Step III} \\ 42 & & \\ \underline{-\ 21} & 1 & \text{Step IV} \\ 21 & & \\ \underline{-\ 21} & \underline{1} & \text{Step V} \\ 0 & 23 & \end{array}$$

The machine "Counts" the number of 21's it subtracts out of 483. Notice it made only 5 subtractions but indicates there are 23 twenty-ones in 483. Why? You are right: It didn't subtract them out 1 at a time. It took 10 twenty-ones out in each of the first two steps.

Actually, if we examine what happened, we find:

483	
- 210	10 twenty-ones
273	
- 210	10 twenty-ones
63	
- 21	1 twenty-one
42	
- 21	11 twenty-ones
21	
- 21	1 twenty-one
	23 twenty-ones

It would be a slow process if we subtract one 21 out each time. Therefore we subtract 21 out in "groups" of ones, tens, hundreds, or as large a group as possible.

Consider the Calculator Method of division in another form. Place the ee number we are "subtracting out" (divisor) on the left and over on the right, how many we subtract out each time.

Example:

58)	1856	
	- 580	10 —— (10 fifty-eights)
	1276	
	- 580	10 —— (10 fifty-eights)
	696	
	- 580	10 —— (10 fifty-eights)
	116	
	- 58	1 —— (1 fifty-eight)
	58	
	- 58	1 —— (1 fifty-eight)
	0	32 —— (32 fifty-eights)

THEN: 1856 ÷ 58 = 32 with r = 0

Example:

124)	15004	
	12400	100
	2604	
	1240	10
	1364	
	1240	10
	124	
	124	1
	0	121

or: 15004 ÷ 124 = 121 with r = 0

Now suppose we examine our regular procedure and compare it to the Calculator Method.

124)	15004
	− 124
	260
	− 248
	124
	− 124

Step I: We state that 124 "goes into" 150 one time and place the 1 above the 0. Actually, this 1 represents how much?

Step II: We say 1 x 124 = and place it under 150 and subtract. How much have we "really" subtracted?

Step III: We "bring down" the next digit (a zero) in the dividend and start repeating Step I. (Now it is 124 into 260.)

How many 124's were subtracted out in each of the three subtractions? Did you first subtract out the "hundreds" of 124's in 15,004, then the tens of 124's and then the number of single (ones) 124's?

Activities

Subtract out and "count"—in ones, tens, hundreds or any size groups you choose—how many times the divisor is contained in the dividend. (Calculator Method)

1.

12) 84

2.

31) 186

3.

32) 1728

4.

24) 3408

5.

123) 17,589

Digits are missing in each of the following division problems. Fill in the missing digits and check your answer.

6.

101) 3232
303
202
202

7.

1001) 123123
1001
2302
2002
3003
3003

8.

$$10001\,\overline{)\,36453645}$$

$\underline{30003}$

64506

$\underline{60006}$

45004

$\underline{40004}$

50005

$\underline{50005}$

The same question can be asked in several different ways. Would restating the same question result in a different answer? Below are four ways of stating a division question. Give the answer to each.

9. 36 X ___ = 180

10. 180 ÷ 36 = ___

11. $\frac{180}{36}$ = ___

12. $36\,\overline{)\,180}$

Below are three forms used to illustrate our division process. Discuss the advantages and disadvantages of each.

13.
$$\begin{array}{rr|r} 54 & \overline{)\,5864} & \\ & \underline{5400} & 100 \\ & 464 & \\ & \underline{432} & \underline{8} \\ r = & 32 & 108 \end{array}$$

14.
$$\begin{array}{rr} 54 & \overline{)\,5864} \\ & \underline{5400} \\ & 464 \\ & \underline{432} \\ r = & 32 \end{array}$$

15.
$$\begin{array}{rl} 54 & \overline{)\,5864} \\ & \underline{54} \\ & \;\;46 \\ & \;\;\underline{\;0} \\ & \;\;464 \\ & \;\;\underline{432} \\ r = & \;\;\;32 \end{array}$$

Translate each verbal problem into a division problem and solve.

16. Platoons, 15 men in each, are to be formed from 3,465 soldiers. How many platoons will there be?

17. For ten years, a man has made equal yearly deposits in a bank. He has totally deposited $36,000.00. How much did he deposit each year?

18. In 8 seconds light travels approximately 1,488,000 miles. About how fast is this per second?

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19. A cable 1860 feet long is to be cut into equal lengths with 21 feet in each length. How many of these lengths will be obtained?

20. If your high school football team has 8 boys who weigh approximately the same, and the boys' total weight is 1,424 lbs., how much does each boy weigh?

21. Every year there is a cheerleader's convention for all cheerleaders throughout the country. This year each school sent 4 girls, combining to make a total of 1,944 cheerleaders in attendance. How many schools participated?

22. Mt. Lenin in Russia is 23,382 feet high. A group of mountainclimbers decided to climb the mountain, and, in order to rest, they made camp at equal levels during their climb. Keeping in mind that there was a camp at the peak, when they reached it, there were 6 camps altogether. How many feet did they climb between each camp?

23. An Olympic swimming pool is divided into lanes for racing. Let's imagine there are 8 lanes in each pool and there are 504 lanes in the state. Arrive at the number of Olympic pools in the state.

24. A. When the Green Bay Packers' and the Baltimore Colts' football teams played each other there were a total of 88 players. If each team had an equal number of players, how many were there on each team?

    B. The stadium in which the two teams played had 60,000 seats, and was divided into 15 equal sections. How many seats were there in each section?

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